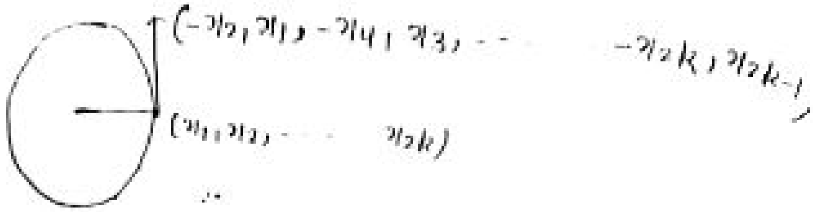
3) let $M = S^n \subseteq \mathbb{R}^{n+1}$ $\quad n = 2k-1 \quad k \geq 1$



$$X_p = \left(-x_2 \frac{\partial}{\partial x^1} + x_1 \frac{\partial}{\partial x^2}\right)(p) + \left(-x_4 \frac{\partial}{\partial x^3} + x_3 \frac{\partial}{\partial x^4}\right)(p) + \cdots$$

$$+ \left(-x_{2k} \frac{\partial}{\partial x^{2k-1}} + x_{2k-1} \frac{\partial}{\partial x^{2k}}\right)(p)$$

thus $X_p \in T_pM \quad \forall\, p \in S^n$

X is nowhere zero vector field on S^n, $n = 2k-1$

* $\mathcal{X}(M)$ denote the set of all vector field on M

let $X, Y \in \mathcal{X}(M)$

$$(X+Y)(p) \equiv X_p + Y_p$$

$$(cX)(p) = cX_p$$

$\mathcal{X}(M)$ is a vector space over $\mathbb{R}$

$C^\infty(M)$: the set of real valued smooth maps on M

if $f \in C^\infty(M)$ and $X \in \mathcal{X}(M)$

Define $(fX)(p) = f(p)X_p \in T_pM$

$$\Rightarrow fX \in \mathcal{X}(M)$$

Define $(Xf)(p) = X_p f \in \mathbb{R}$

$$Xf : M \longrightarrow \mathbb{R} \quad \Rightarrow Xf \in C^\infty(M)$$

Definition :- For all $X, Y \in \mathfrak{X}(M)$ $f \in C^\infty(M)$ and $p \in M$

$[X,Y](p)(f) = X_p(Yf) - Y_p(Xf)$

$[X,Y]$ is called Lie Bracket of X and Y

otherwise

$[X,Y](f) = X(Yf) - Y(Xf)$

check

(i) $[X,Y](af+bg) = a[X,Y]f + b[X,Y]g$

(ii) $[X,Y](fg) = f[X,Y](g) + g[X,Y](f)$

(i) $[X,Y](af+bg) = X(Y(af+bg)) - Y(X(af+bg))$

$= X(aYf + bYg) - Y(aXf + bXg)$

$= aX(Yf) - aY(Xf) + bX(Yg) - bY(Xg)$

$= a[X(Yf) - Y(Xf)] + b[X(Yg) - Y(Xg)]$

$= a[X,Y](f) + b[X,Y](g)$

(ii) $[X,Y](fg) = X(Yfg) - Y(Xfg)$

$= X[g(Yf) + f(Yg)] - Y[g(Xf) + f(Xg)]$

$= f[X,Y](g) + g[X,Y](f)$

Ex: ① let $M = \mathbb{R}^2$ $X = x\frac{\partial}{\partial y}$ $Y = y\frac{\partial}{\partial y}$

$[X,Y] = ?$

Soln let $f \in C^\infty(\mathbb{R}^2)$

$[X,Y](f) = X(Yf) - Y(Xf)$

$= X\left(y\frac{\partial f}{\partial y}\right) - Y\left(x\frac{\partial f}{\partial y}\right)$

$$[x,y](f) = x\frac{\partial}{\partial y}\left(y\frac{\partial f}{\partial y}\right) - y\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial y}\right)$$

$$= x\left[1\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2}\right] - yx\frac{\partial^2 f}{\partial y^2}$$

$$[x,y](f) = x\frac{\partial f}{\partial y}$$

$$\boxed{[x,y] = x\frac{\partial}{\partial y} = X}$$

<u>Theorem</u> let $X, Y, Z \in \mathcal{X}(m)$, $f, g \in C^\infty(m)$, $a, b \in \mathbb{R}$ then

(i) $[X,Y] = -[Y,X]$ (anti commutative)

(ii) $[X,X] = 0$

(iii) $[aX+bY, Z] = a[X,Z] + b[Y,Z]$

or

$[X, aY+bZ] = a[X,Y] + b[X,Z]$

(iv) $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

(v) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi identity)

<u>Proof</u> (i) let $h \in C^\infty(m)$

$$[X,Y](h) = X(Yh) - Y(Xh)$$

$$= -\{Y(Xh) - X(Yh)\}$$

$$= -[Y,X](h)$$

Thus $[X,Y] = -[Y,X]$.

$\{then$ (i) $Y = X$

then $[X, X] = 0$

$$[aX+bY, Z](h) = (aX+bY)(Zh) - Z((aX+bY)h)$$
$$= aX(Zh) + bY(Zh) - aZ(Xh) - bZ(Yh)$$
$$= a[X, Z](h) + b[Y, Z](h)$$

thus $[aX+bY, Z] = a[X, Z] + b[Y, Z]$

similarly

$$[X, aY+bZ] = a[X, Y] + b[X, Z]$$

(ii) let $h \in C^{\infty}(M)$

then $[fX, gY](h) = (fX)(gY)(h) - (gY)(fX)(h)$

$$= (fX)(g(Yh)) - (gY)(f(Xh))$$
$$= f\{gX(Yh) + (Yh)(Xg)\} - g\{fY(Xh) + (Xh)(Yf)\}$$
$$= f(Yh)(Xg) + fgX(Yh) - g(Xh)(Yf) - gfY(Xh)$$
$$= fg\{X(Yh) - Y(Xh)\} + f(Xg)(Yh) - g(Yf)(Xh)$$
$$= fg[X, Y](h) + f(Xg)(Y)(h) - g(Yf)(X)(h)$$

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

(iii) let $f \in C^{\infty}(M)$

$$[X, [Y, Z]](f) = X[Y, Z](f) - [Y, Z](Xf)$$
$$= X\{Y(Zf) - Z(Yf)\} - Y(ZXf) + Z(YXf)$$
$$= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf)) \quad \text{--- (1)}$$

similarly

$$[y,[z,x]] = y(z(xf)) - y(x(zf)) - z(x(yf)) + x(z(yf))$$ — (ii)

$$[z,[x,y]] = z(x(yf)) - z(y(xf)) - x(y(zf)) + y(x(zf))$$ — (iii)

adding (i), (ii) and (iii) we get

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = x(y(zf)) - x(z(yf)) - y(z(xf))$$
$$+ z(y(xf)) + y(z(xf)) - y(x(zf)) - z(x(yf)) + x(z(yf))$$
$$+ z(x(yf)) - z(y(xf)) - x(y(zf)) + y(x(zf))$$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$$

ex: let M be smooth manifold at (U,ϕ) be local chart on M with co-ordinate $x^1, x^2, \ldots\ldots x^n$ then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0 \qquad i,j = 1,2,\ldots\ldots n$$

Proof It is sufficient to show that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_{(p)}(f) = 0 \qquad \forall f \in C^\infty(U)$$
$$p \in U$$

let $u_1, u_2, \ldots\ldots\ldots u_n$ be usual coordinates of $\mathbb{R}^n$ then

$$\frac{\partial}{\partial x^i}\Big|_{(p)}(f) = \frac{\partial}{\partial u_i}(f\circ\phi^{-1})\circ\phi_{(p)}$$

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$$\left.\frac{\partial}{\partial x^j}\right|_{p}(f) = \left.\frac{\partial}{\partial x^i}\right|_{p}\left(\frac{\partial}{\partial x^j}(f)\right) - \left.\frac{\partial}{\partial x^j}\right|_{p}\left(\frac{\partial}{\partial x^i}(f)\right)$$

$$= \left.\frac{\partial}{\partial x^i}\right|_{p}\left(\frac{\partial}{\partial u_j}(f\circ\phi^{-1})\circ\phi\right) - \left.\frac{\partial}{\partial x^j}\right|_{p}\left(\frac{\partial}{\partial u_i}(f\circ\phi^{-1})\circ\phi\right)$$

$$\frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}(f\circ\phi^{-1})\circ\phi\circ\phi^{-1}\right)_{\phi(p)} - \frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial u_i}(f\circ\phi^{-1})\circ\phi\circ\phi^{-1}\right)_{\phi(p)}$$

$$= \frac{\partial^2}{\partial u_i \partial u_j}$$

$$= \left\{\frac{\partial^2}{\partial u_i \partial u_j}(f\circ\phi^{-1}) - \frac{\partial^2}{\partial u_j \partial u_i}(f\circ\phi^{-1})\right\}\phi(p)$$

$= 0$ as $f \in C^{\infty}(U)$

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0 \quad \forall\ i, j = 1, 2, \ldots\ n$$

Ex-(1) If X, Y are smooth vector field on $\mathbb{R}^3$ defined by

$$X = \frac{\partial}{\partial x^1} \qquad Y = \frac{\partial}{\partial x^2} + e^{x^1}\frac{\partial}{\partial x^3}$$

compute $[X, Y]$

Solⁿ let f be a smooth real valued map on $\mathbb{R}^3$ then

$$[X, Y](f) = X(Yf) - Y(Xf)$$

$$= X\left[\frac{\partial f}{\partial x^2} + e^{x^1}\frac{\partial f}{\partial x^3}\right] - Y\left\{\frac{\partial f}{\partial x^1}\right\}$$

$$= \frac{\partial}{\partial x^1}\left[\frac{\partial f}{\partial x^2} + e^{x^1}\frac{\partial f}{\partial x^3}\right] - \left(\frac{\partial}{\partial x^2} + e^{x^1}\frac{\partial}{\partial x^3}\right)\left(\frac{\partial f}{\partial x^1}\right)$$

$$= \cancel{\frac{\partial^2 f}{\partial x^1 \partial x^2}} + e^{x^1}\frac{\partial f}{\partial x^3} + \cancel{e^{x^1}\frac{\partial^2 f}{\partial x^1 \partial x^3}} - \cancel{\frac{\partial^2 f}{\partial x^2 \partial x^1}} - e^{x^1}\frac{\partial^2 f}{\partial x^3 \partial x^1}$$

$$e^{x^1}\frac{\partial}{\partial x^3}$$

$$\boxed{[X,Y] = e^{x^1}\frac{\partial}{\partial x^3}}$$

Q. If X and Y are vector field on $\mathbb{R}^2$ given by

$$X = (x^1)^2\frac{\partial}{\partial x^1} \qquad Y = \frac{\partial}{\partial x^1} + (x^2)^2\frac{\partial}{\partial x^2}$$

compute $[X,Y]$

Solⁿ let f be smooth real valued map defined on $\mathbb{R}^2$

$$[X,Y](f) = X(Yf) - Y(Xf)$$

$$= X\left[\frac{\partial f}{\partial x^1} + (x^2)^2\frac{\partial f}{\partial x^2}\right] - Y\left((x^1)^2\frac{\partial f}{\partial x^1}\right)$$

$$= \left((x^1)^2\frac{\partial}{\partial x^1}\right)\left(\frac{\partial f}{\partial x^1} + (x^2)^2\frac{\partial f}{\partial x^2}\right) - \left(\frac{\partial}{\partial x^1} + (x^2)^2\frac{\partial}{\partial x^2}\right)\left((x^1)^2\frac{\partial f}{\partial x^1}\right)$$

$$= \cancel{(x^1)^2\frac{\partial^2 f}{\partial x^{1^2}}} + (x^1)^2(x^2)^2\frac{\partial^2 f}{\partial x^1\partial x^2} - 2x^1\frac{\partial f}{\partial x^1} - \cancel{(x^1)^2\frac{\partial^2 f}{\partial x^{1^2}}}$$
$$- (x^1)^2(x^2)^2\frac{\partial^2 f}{\partial x^1\partial x^2}$$

$$= -2x^1\frac{\partial f}{\partial x^1}$$

$$\boxed{[X,Y] = -2x^1\frac{\partial}{\partial x^1}}$$

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Q: let M be smooth n-dimensional manifold (U,ϕ) be a local chart on M with coordinate $x^1, x^2, \ldots\ldots, x^n$

If $X = \sum f_i \frac{\partial}{\partial x^i}$ and $Y = \sum g_j \frac{\partial}{\partial x^j}$

are smooth vector field on M

then show that

$$[X,Y] = \sum_j \left(\sum_i \left(f_i \frac{\partial g_j}{\partial x^i} - g_i \frac{\partial f_j}{\partial x^i} \right) \right) \frac{\partial}{\partial x^j}$$

Sol^n let $h \in C^\infty(M)$

$$[X,Y](h) = X(Yh) - Y(Xh)$$

$$= X\left(\sum_j g_j \frac{\partial h}{\partial x^j} \right) - Y\left(\sum_i f_i \frac{\partial h}{\partial x^i} \right)$$

$$= \sum_i f_i \frac{\partial}{\partial x^i} \left(\sum_j g_j \frac{\partial h}{\partial x^j} \right) - \sum_j g_j \frac{\partial}{\partial x^j} \left(\sum_i f_i \frac{\partial h}{\partial x^i} \right)$$

$$= \sum_j \left[\sum_i \left(f_i \frac{\partial g_j}{\partial x^i} \frac{\partial h}{\partial x^j} + \cancel{f_i g_j \frac{\partial^2 h}{\partial x^i \partial x^j}} - g_i \frac{\partial f_j}{\partial x^i} \frac{\partial h}{\partial x^j} - \cancel{f_i g_j \frac{\partial^2 h}{\partial x^i \partial x^j}} \right) \right]$$

$$= \sum_j \left[\sum_i \left(f_i \frac{\partial g_j}{\partial x^i} - g_i \frac{\partial f_j}{\partial x^i} \right) \right] \frac{\partial h}{\partial x^j}$$

(1) Lie Algebra:- let V be a vector space over $\mathbb{R}$. A Lie algebra is a pair $(V, [\ ])$ where $[\ ]$ is bracket operation which is $\mathbb{R}$ bilinear map of $V \times V \longrightarrow V$ satisfying anti commutativity condition and jacobi identity

i.e (i) $[x,y] = -[y,x]$

(ii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall\ x,y,z \in V$

ex:(1) let V be any vector space over $\mathbb{R}$

Define $[\ ] : V \times V \longrightarrow V$ as

$[x,y] = 0 \quad \forall\ x,y \in V$

Then $[\ ]$ is anti commutative and satisfy jacobi identity i.e $(V, [\ ])$ is a Lie algebra.

Every real vector space is Lie-algebra.

ex-2 consider space $M(n,\mathbb{R})$ i.e the space of $n \times n$ matrices over $\mathbb{R}$

Define $[A,B] = AB - BA \quad \forall\ A,B \in M(n,\mathbb{R})$

Then $[\ ]$ is anti commutative and satisfy jacobi identity So $(M(n,\mathbb{R}), [\ ])$ is a Lie algebra.

Lie differentiation:- let M be smooth manifold at $\mathcal{X}(M)$ denoted by set of smooth vector field on M. Then linear map

$L_X : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$

given by $L_X(Y) = [X,Y] \quad \forall\ X,Y \in \mathcal{X}(M)$

called Lie differentiation of Y w.r.t. X (2)

Definition :- ϕ-related vector field.

The vector field $X \in \mathcal{U}(M)$ and $Y \in \mathcal{U}(N)$ are said to be ϕ-related if

$$D\phi(p)(X(p)) = Y\phi(p) \quad \forall \; p \in M$$

i.e. $D\phi \circ X = Y \circ \phi$

let $g \in C^{\infty}(N)$

then $X \in \mathcal{U}(M)$ and $Y \in \mathcal{U}(N)$ are ϕ-related iff

$$D\phi(p)X(p)(g) = Y(\phi(p))(g)$$

$$X(p)(g \circ \phi) = (Yg)(\phi(p))$$

i.e. $X(g \circ \phi)(p) = (Yg \circ \phi)(p)$

$$X(g \circ \phi) = Yg \circ \phi \quad \forall \; p \in M$$

//

Theorem If $X_i \in \mathcal{U}(M)$ is ϕ-related to $Y_i \in \mathcal{U}(N)$ for $i = 1, 2$

then $[X_1, X_2]$ is ϕ-related to $[Y_1, Y_2]$

Proof let $g \in C^{\infty}(N)$ then we shall show that

$$[X_1, X_2](g \circ \phi) = [Y_1, Y_2](g) \circ \phi$$

consider $X_1 X_2 (g \circ \phi) = X_1 (Y_2 g \circ \phi)$

as X_2 is ϕ-related Y_2

$= Y_1(Y_2 g) \circ \phi$ as X_1 is ϕ-related to Y_1

$= Y_1 Y_2 (g) \circ \phi$ ——— (1)

Similarly we have

$$X_2X_1(g\circ\phi) = Y_2Y_1(g)\circ\phi \quad \text{——— (2)}$$

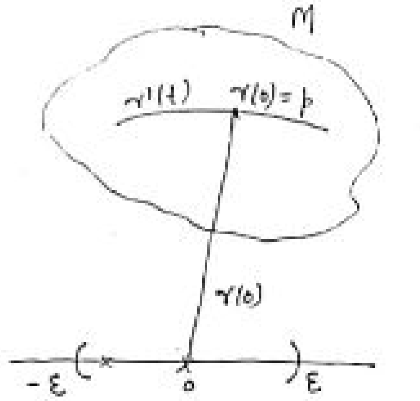
Hence from (1) and (2) we have

$$[X_1,X_2](g\circ\phi) = [Y_1,Y_2](g)\circ\phi$$

$\Rightarrow$ $[X_1,X_2]$ is ϕ-related to $[Y_1,Y_2]$

Definition :- let M be a smooth manifold and $X\in \mathcal{X}(M)$ and $p\in M$. A smooth curve γ is said to be an integral curve of X passing through p if $\gamma(0)=p$

$$\gamma'(t) = X(\gamma(t)) = X_{\gamma(t)} \quad \text{for } t \text{ is domain of}$$



$$\gamma : (-\epsilon,\epsilon) \xrightarrow{\text{smooth}} M$$

let (U,ϕ) be a local chart arround p with local coordinate x^i

then
$$\underset{T_{\gamma(t)}M}{\gamma'(t)} = X(\gamma(t)) = \sum_{i=1}^{m} X(x^i)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \quad \text{——— (1)}$$

where $m=\dim M$

$\forall\ t\in$ domain of γ

By local representation of derivative of the smooth maps we have

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{d}{dt}(x^i \circ \gamma)\Big|_{t=t_0} \frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)} \quad \text{— (2)}$$

from (1) and (2) we have

$$\frac{d}{dt}(x^i \circ \gamma) = X(x^i)$$

and $x^i(p) = 0 \quad i = 1, 2, \dots n$

By existance of uniqueness theorem for O.D.E the above system of linear equation has unique solution.

ex:- let $M = \mathbb{R}^2$ and $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $p = (a, b)$

find integral curve of X passing through p

Soln If $\gamma = \gamma(x(t), y(t))$ be an integral curve of X $t \in (-\epsilon, \epsilon)$

Then $\frac{dx}{dt} = -y \qquad x(0) = a$

$\frac{dy}{dt} = x \qquad y(0) = b$

$$\frac{d^2x}{dt^2} = -\frac{dy}{dt} = -x$$

$$\frac{d^2x}{dt^2} + x = 0$$

$x = c_1 \cos t + c_2 \sin t$

$x(0) = a \Rightarrow c_1 = a$

$\cancel{\frac{d^2y}{dt^2} = \frac{dx}{dt} = y}$ $\qquad \frac{dx}{dt} = -c_1 \sin t + c_2 \cos t$

$\cancel{\frac{d^2y}{dt^2} + y = 0}$ $\qquad -y = -c_1 \sin t + c_2 \cos t$

$y = c_2 \cos t - c_1 \sin t$

$\cancel{y = c_1 \cos t + c_2 \sin t}$ $\quad y(0) = b \Rightarrow c_2 = b$

$x(t) = a \cos t + b \sin t$

$y(t) = b \cos t - a \sin t$

$p = (a,b) \neq (0,0)$

<u>Distribution</u> :- let M be smooth manifold. A k-dimension ($k \geq 0$) dimension of distribution D on M is a map which associate to each point $p \in M$ a k-dimensional subspace D_p of corresponding tangent space T_pM

$$D : p \longrightarrow D_p \subseteq T_pM$$

ex-① let M be a n-dimensional smooth manifold then $p \longrightarrow T_pM \ \forall \ p \in M$ is an n-dimensional distribution on M

we say that a vector field X on M.

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consider $M = \mathbb{R}^3$ with standard atlas and coordinates x^1, x^2, x^3.

let $X = \frac{\partial}{\partial x^1}$ and $Y = \frac{\partial}{\partial x^2} + \exp(x^1)\frac{\partial}{\partial x^3}$ then these

two vector field generate a 2 dimensional distribution

D on $\mathbb{R}^3$

$D = \langle X, Y \rangle$

ex: If X is a global non vanishing vector field on $\mathbb{R}^3$

then it determines a 1-dimensional distribution $\mathbb{R}^3$

Definition :- A K-dimensional distribution D on M is said to differentiable (i.e C^∞) iff for $p \in M$

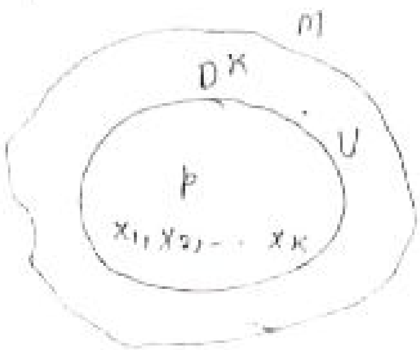
there exist a nbd U of p and C^∞ vector member

$X_1, X_2, \ldots\ldots X_K$ on U such that for each point

$q \in U$

$X_1(q), X_2(q), \ldots\ldots\ldots X_K(q)$ form a basis for subspace

$D_q \subseteq T_q M$



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Definition :- A submanifold N is said to be integral submanifold at p for the distribution if $p \in N$ and $D_q = T_q N \ \forall q \in N$. D is integrable if for any point $p \in M$ there is an integral submanifold N at p.

Definition :- A k-dimensional distribution D on a smooth manifold is said to be involutive if $X, Y \in D$ $\Rightarrow [X, Y] \in D$

Frobenius theorem :- A k-dimensional distribution on a smooth manifold is integrable iff it is involutive.

Ex: consider $M = \mathbb{R}^3$ with standard atlas on M and coordinate (x^1, x^2, x^3)

let $X = \partial/\partial x^1$ $\quad Y = \partial/\partial x^2 + \exp(x^1) \dfrac{\partial}{\partial x^3}$

let D be a distribution on $\mathbb{R}^3 (= M)$ generated by X and Y

$$[X, Y](f) = X(Yf) - Y(Xf)$$

$$= X\left[\frac{\partial f}{\partial x^2} + \exp(x^1)\frac{\partial f}{\partial x^3}\right] - Y\left[\frac{\partial f}{\partial x^1}\right]$$

$$= \frac{\partial}{\partial x^1}\left[\frac{\partial f}{\partial x^2} + \exp(x^1)\frac{\partial f}{\partial x^3}\right] - \left[\frac{\partial}{\partial x^2} + \exp(x^1)\frac{\partial}{\partial x^3}\right]\left[\frac{\partial f}{\partial x^1}\right]$$

$$= \cancel{\frac{\partial^2 f}{\partial x^1 \partial x^2}} + \exp(x^1)\frac{\partial f}{\partial x^3} + \exp(x^1)\frac{\partial^2 f}{\partial x^1 \partial x^3} - \frac{\partial^2 f}{\partial x^2 \partial x^1}$$

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$[x,y] = \exp(x^1)\frac{\partial}{\partial x^3}$ which is not linear combination of x and y so $[x,y] \notin D$

Hence it is not involutive and hence distribution is not integrable.

Tangent space: let M be m-dimensional smooth manifold at $p \in M$. let T_pM be tangent space to M at p which is an m-dimensional vector space over $\mathbb{R}$

Cotangent space:- let M be m-dimensional smooth manifold and $p \in M$. The dual space of T_pM is denoted by T_p^*M is called cotangent space to M at the point of M

The element of T_p^*M is called covector at $p \in M$

let $f \in C^\infty(M)$ and $p \in M$ Define

$$(df)(p) : T_pM \longrightarrow \mathbb{R} \text{ as}$$

$$(df)_{(p)}(X_p) = X_p f$$

then $(df)_{(p)}$ is linear so $df(p) \in T_p^*M$

let (U, ϕ) be a local chart on M with coordinate $x^1, x^2, \ldots\ldots\ldots x^m$ at $p \in U$ then standard basis for T_pM is [illegible] $\left.\frac{\partial}{\partial x^2}\right|_p, \left.\frac{\partial}{\partial x^3}\right|_p, \ldots \left.\frac{\partial}{\partial x^n}\right|_p\Big\}$

[illegible] basis for T_p^*M

$dx^i(p) : T_pM \longrightarrow IR$

as $$dx^i(p)\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left(\frac{\partial x^i}{\partial x^j}\right)_p = \delta^i_j$$

Then $$B^* = \{dx^1(p), dx^2(p), \ldots\ldots dx^n(p)\}$$

is basis for T_p^*M which is dual to B

Tangent bundle :- let M be a smooth n-dimensional manifold

let
$$TM = \bigcup_{p \in M} T_pM$$

consider a map
$$\pi : TM \longrightarrow M \text{ as}$$
$$\pi(v) = p \text{ if } v \in T_pM$$

we shall show that TM admits smooth structure

let us take a smooth atlas on M let (U, ϕ) be a local chart on above atlas

let $\hat{U} = \pi^{-1}(U)$

$$\pi : \underset{\pi^{-1}(U)}{TM} \longrightarrow \underset{U}{M}$$

$\hat{U} = \pi^{-1}(U) = \bigcup_{p \in U} T_p M$ which is an open subset

us define a map

$\hat{\phi} : \hat{U} \longrightarrow \phi(U) \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m \cong \mathbb{R}^{2m}$

$\hat{\phi}(v) = \left(x^1(\pi(v)), x^2(\pi(v)), \ldots x^m(\pi(v)), v(x^1), \ldots v(x^m) \right)$

$\longrightarrow \phi(U) \times \mathbb{R}^m$ is bijective

$\hat{\phi}$ is also bijective

$(\hat{U}, \hat{\phi})$ is local chart on TM

$v = \left(\sum_{i=1}^{m} v_i \frac{\partial}{\partial x^i}\Big|_p \right)$

$v(x^j) = \sum_{i=1}^{m} v_i \frac{\partial x^j}{\partial x^i}\Big|_p$

$v(x^j) = \sum_{i=1}^{m} v_i \delta_{ij}$ $\quad i = j \quad \delta_{ij} = 1$

$i \neq j \quad \delta_{ij} = 0$

$v(x^j) = v_j$

us define coordinate function

ξ_j on $\hat{U}$ as

$\xi_j(v) = u_j \circ \hat{\phi}(v) \qquad 1 \le j \le 2m$

$$\xi_j(v) = \begin{cases} x^j(\pi(v)) & 1 \le j \le m \\ v(x^j) & m+1 \le j \le 2m \end{cases}$$

let us consider a collection

$\{ (\hat{U}, \hat{\phi}) : \text{ is local chart on } M \}$

shall show that above collection forms a smooth atlas on

it suffices that if two charts of above collection

their transition maps are smooth

let $(\hat{U}, \hat{\phi})$ and $(\hat{V}, \hat{\psi})$ be two charts on above collection

then these charts overlap iff (U, ϕ) and (V, ψ) overlap

(7) Let x^i and y^i be two local coordinates on U and V respectively, $1 \le i \le n$

we shall show that

$$\hat{\phi} \circ \hat{\psi}^{-1} : \hat{\psi}(\hat{U} \cap \hat{V}) \longrightarrow \hat{\phi}(\hat{U} \cap \hat{V}) \quad \text{is smooth}$$

$$\hat{\psi} : \hat{V} \longrightarrow \psi(V) \times \mathbb{R}^n$$
$$v \longrightarrow (a, b) \qquad a \in \psi(V), \; b \in \mathbb{R}^n$$

let $(a,b) \in \psi(U \cap V) \times \mathbb{R}^n$

then $$\hat{\psi}^{-1}(a,b) = \sum_{i=1}^{n} b^i \frac{\partial}{\partial y^i}\Big|_{\psi^{-1}(a)}$$

where $b = (b^1, b^2, \ldots \ldots b^n) \in \mathbb{R}^n$

To show

$\hat{\phi} \circ \hat{\psi}^{-1} : \hat{\psi}(U \cap V) \longrightarrow \hat{\phi}(U \cap V)$ is smooth, we shall show

that $u_i \circ \hat{\phi} \circ \hat{\psi}^{-1}$ is smooth $1 \le i \le 2n$

case(i) let $1 \le i \le n$

then

$$u_i \circ \hat{\phi} \circ \hat{\psi}^{-1}(a,b) = x^i \circ (\pi \circ \psi^{-1}(a,b))$$
$$= x^i \; x^i \psi^{-1}(a)$$

$$(u_1 \circ \hat{\phi} \circ \hat{\psi}^{-1}(a,b), \; u_2 \circ \hat{\phi} \circ \hat{\psi}^{-1}(a,b), \ldots \ldots u_n \circ \hat{\phi} \circ \hat{\psi}^{-1}(a,b))$$
$$= (x^1(\psi^{-1}(a)), x^2 \psi^{-1}(a) \ldots \ldots \ldots x^n \psi^{-1}(a))$$
$$= \phi \circ \psi^{-1}(a) \text{ is smooth}$$

Hence $(u_1 \circ \hat{\phi} \circ \hat{\psi}^{-1}, \ldots \ldots \ldots u_n \circ \hat{\phi} \circ \hat{\psi}^{-1})$ is also smooth

$y^j = n + r$ with $1 \le r \le n$

$$\hat{\phi}\circ\hat{\psi}^{-1}(a,b) = U_j\circ\hat{\phi}\left(\sum_{i=1}^{m} b^i \frac{\partial}{\partial y^i}\Big|\psi^{-1}(a)\right)$$

$$= \sum_{i=1}^{m} b^i \frac{\partial}{\partial y^i}\Big|_{\psi^{-1}(a)} (x^r)$$

$$= \sum_{i=1}^{m} b^i \frac{\partial x^r}{\partial y^i}\Big|\psi^{-1}(a)$$

which are a components of Jacobian of the map $\phi\circ\psi^{-1}$, but $\phi\circ\psi^{-1}$ is smooth so $U_j\circ\hat{\phi}\circ\hat{\psi}^{-1}$ is smooth $j = n+r$ with $1 \le r \le n$

Thus $\hat{\phi}\circ\hat{\psi}^{-1}$ is smooth

Similarly we can show that $\hat{\psi}\circ\hat{\phi}^{-1}$ is also smooth so

$\{(\hat{U},\hat{\phi}),(\hat{V},\hat{\psi})$ is local chart on $M\}$ determined a smooth structure on TM

so that it is a smooth manifold of dimension $2n$.

Cotangent bundle :-

Define $\hat{\phi} : \hat{U} \longrightarrow \phi(U)\times\mathbb{R}^m$

$$T^*M = \bigcup_{p\in M} T_p^*M$$

$$\pi : T^*M \longrightarrow M$$

$$\pi(\tau) = p \quad \text{if } \tau \in T_p^*M$$

let (U,ϕ) be local chart

$$\hat{U} = \pi^{-1}(U)$$

$$\hat{\phi}(\tau) = \left(x^1\pi(\tau), \ldots, x^m\pi(\tau), \tau\left(\frac{\partial}{\partial x^1}\Big|_p\right), \ldots, \tau\left(\frac{\partial}{\partial x^m}\Big|_p\right)\right)$$

$\tau = \tau_i\, dx^i|_p$

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